QP Code: 22/PT/13/IXA(i)

POST-GRADUATE COURSE

Term End Examination — June, 2022/December, 2022

MATHEMATICS

Paper-9A(i) : ADVANCED COMPLEX ANALYSIS

(Pure Mathematics)

(Spl. Paper)

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

(Symbols have their usual meanings)

Answer Question No. **1** and any *four* from the rest :

1. Answer any *five* questions : $2 \times 5 = 10$

a) Define subharmonic and superharmonic function.

b) Find the order of e^{z^2}

c) Find the exponent of convergence of the zeros of cos z.

d) Find the branch points of $f(z) = \left(z^3 - 1\right)^{1/2}$.

e) Show that a rational function is meromorphic.

f) Given that the identity $\sin^2 z + \cos^2 z = 1$ holds for real values of z. Prove that it also holds for all complex values of z.

g) Find the value of the infinite product $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right) \ldots.$

2. a) State and prove Poisson's integral formula for harmonic function.

b) Let f be a function regular in the closed disc $|z| \leq R$ and let $u(r, \theta)$ be its real part. If $u(r, \theta) \geq 0$, then prove that

$$\frac{R-r}{R+r}\, u(0, 0) \leq u(r, \theta) \leq \frac{R+r}{R-r}\, u(0, 0),$$

where $0 \leq r \leq R$. 6 + 4

3. a) Let $f(z)$ be analytic at the point z_0. If $f'(z_0)=0$, $f''(z_0)=0, \ldots\ldots,$ $f^{(k-1)}(z_0)=0$ but $f^{(k)}(z_0)\neq 0$, then show that the transformation $\omega=f(z)$ magnifies angles at z_0 by *k*-times.

   b) Prove that an absolutely convergent infinite product is always convergent. 6 + 4

4. a) State and prove Hadamard's three circles theorem.

   b) If $f(z)$ is an entire function of finite order ρ, then show that $n(r)=O(r^{\rho+\epsilon})$ for $\epsilon>0$ and for sufficiently large values of *r*. 6 + 4

5. a) State and prove Weierstrass factorization theorem.

   b) Show that $\int_0^\infty \frac{x^{\alpha-1}}{1+x}\, dx=\frac{\pi}{\sin\pi\alpha}$, given that $0 < \alpha < 1$. 5 + 5

6. a) Let $u(x, y) \not\equiv$ constant be harmonic in a domain *D*. Prove that $u(x,y)$ has neither a maximum nor a minimum at any point of *D*.

   b) Show that every polynomial is of order zero.

   c) Show that the series $\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$ and $\sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$ are direct analytic continuation of each other. 4 + 3 + 3

7. a) Find a Schwarz-Christoffel transformation that maps the upper half plane to the inside of the triangle with vertices – 1, 1 and $i\sqrt{3}$.

   b) Obtain the residue of the Gamma function at its singularities.

   c) Discuss the convergence or divergence of the infinite product $\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$. 4 + 3 + 3